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A COOPERATIVE STUDY OF ONE-COMMODITY MARKET GAMES

by
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Abstract: *A competitive model is attached to a particular market where several agents trade a same commodity. We focus on the stability of the price systems and define a solution concept, akin to the core, proving an existence result.* (JEL: *C71, D40, D43*)

Keywords*:* *one-commodity market, cooperative games, price systems*

1. *Introduction*

Cooperative game theory has been frequently used to model various economic problems. As is well-known, economic systems were treated as cooperative games, and the core of an economy was revealed as the main solution concept. But also some specific economic problems are modeled in a cooperative competitive framework. Shapley and Shubik (1969) derived the "market game" from an exchange economy. "Production games" are used to study production problems in a competitive environment (see Owen, 1975). Cost allocation problems are also modeled as cooperative games by several authors (see Bendali et al., 2001; Granot and Huberman, 1984; Gambarelli, 1997).

Our approach concerns a particular market with several agents who trade one commodity. Each bundle of commodity sold by an agent has a social value which includes all investiments, specific costs and the expected profit.

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The agent should trade his bundle of commodity at a price which allows him to cover this value. We focus on the problem of stability of prices when these are negotiated by coalitions of agents.

2. *The Model*

In the sequel we will adopt the following notation and terminology:

- If $x, y \in R^n$ we write $x \geq y$ ($x > y$), if $x_i \geq y_i$ ($x_i > y_i$), for all $i = 1, 2, \ldots, n$.
- If $x \in R^n$ and $\varnothing \neq S \subseteq \{1, 2, \ldots, n\}$, $\hat{x}_S$ is the n-vector whose i-th component is x_i if $i \in S$, and 0 if $i \notin S$ and x_s is the $|S|$-dimensional vector $(x_i)_{i \in S}$ (for $S = \{i\}$, simply write $\hat{x}_i$, respectively, x_i).
- The complement of S with respect to $N = \{1, 2, \ldots, n\}$ will be denoted by $\overline{S}$, and the generic symbol for any family of subsets of N is **S**.
- A family **S** is said to be balanced if there exists a system of positive weights $(w_S)_{S \in \mathbf{S}}$ such that $\forall i \in N, \sum_{S \in \mathbf{S}, i \in S} w_S = 1$.
- A scalar function $f : R^n \to R$ is said to be increasing if $x \geq y \Rightarrow f(x) \geq f(y)$, and strictly increasing if $x \geq y, x \neq y \Rightarrow f(x) > f(y)$.

We want to consider the real situation of a market where several agents trade the same type of commodity. Each agent sells a bundle of commodity demanded by the consumers. In principle, each consumer has his own preferred agent, but he can change him if the prices of other agents become more attractive. Therefore, each agent should satisfy a demand which is depending on all prices. Each group of agents (coalition) negociates the individual prices of its member in order to equilibrate the value of the good sold with the total amount of money the consumers should pay for it.

Let us introduce our model:

- $N = \{1, 2, \ldots, n\}$ denotes the set of agents;
- $p \in R^n_+$ is a system of prices when $p = (p_1, p_2, \ldots, p_n)$, where p_i is the price agent i charges for one unit of the commodity;

– $d : R_+^n \to R_+^n$ is a demand function, $d = (d_1, d_2, \ldots, d_n)$, where $d_i(p)$ represents the amount of the commodity that is purchased from the i-th agent, when the price system is p;
– $v : R_+^n \to R$ denotes the value function. If the total demand is represented by a non-negative vector x, then its value on the market which includes all investiments, specific costs and expected profit is $v(x)$. In particular, if $S \subset N$, then $v(\hat{x}_S)$ is the value of the demand of the agents in S.

The triple $G = (N, d, v)$ is called *one-commodity market game*.

In this context a stable system of prices $p^* \in R_+^n$ can be defined as satisfying the following conditions:

(1) $$\langle p^*, d(p^*) \rangle = v(d(p^*))$$

There are no $p \in R_+^n$ and $\varnothing \neq S \subseteq N$ such that

(2.a) $$p_S < p_S^*,\ p_{\bar{S}} = p_{\bar{S}}^*,$$

(2.b) $$\langle p_S, d_S(p) \rangle \geq v(\hat{d}_S(p))$$

Condition (1) simply says that the total revenue equals the value of the total demand at the price p^*. By condition (2), p^* is stable; no coalition of agents can become more attractive for the consumers, reducing the prices, but exceeding at the same time, the value of the good sold.

Definition 1. *The pseudo-core of a one-commodity market $G = (N, d, v)$ is the set of price systems satisfying conditions* (*1*) *and* (*2*)*. It will be denoted by* $PC(G)$.

The market-type considered is included in the frame of the imperfect competition. In fact, we know that the agents who operate in this economic framework can decide about the price of the commodity. But when all agents find an agreement by which it is possible to get a stability of the market, then we can introduce the concept of stable system of prices satisfying the above conditions (1) and (2.a-2.b). In this case, when these conditions are satisfied the market considered moves to an equilibrium.

Example 1. Let us consider a *Cournot-Bertrand oligopoly,* where the agent i sells the good at the price p_i, $i = 1, 2, \ldots, n$. The i-th agent demand d_i is depending on the price system $p = (p_1, p_2, \ldots, p_n)$, and its cost function (including a reasonable profit), $c_i \cdot d_i(p)$, where $c_i > 0$ is linear with respect to the demand.

According to our terminology (N, d, v) is a *one-commodity market game*, where $v : R_+^n \to R$ is defined by $v(x) = \sum_{i \in N} c_i \cdot x_i$.

From Definition 1 it follows that $p^* = (p_1^*, p_2^* \ldots, p_n^*)$ is a stable system of prices if

$$\sum_{i \in N} p_i^* \cdot d_i(p^*) = \sum_{i \in N} c_i \cdot d_i(p^*)$$

and

$$S \subset N,\ p_S < p_S^* \Rightarrow \sum_{i \in S} p_i \cdot d_i(p_S, p_{\overline{S}}^*) < \sum_{i \in S} c_i \cdot d_i(p_S, p_{\overline{S}}^*)$$

In particular, it follows that

$$\forall i \in N,\ p_i < p_i^* \Rightarrow p_i \cdot d_i\left(p_i, p_{\overline{\{i\}}}^*\right) < c_i \cdot d_i\left(p_i, p_{\overline{\{i\}}}^*\right)$$

Assuming that the demand functions are strictly positive, it easily results that $p^* = (c_1, c_2 \ldots, c_n)$ is a stable system of prices.

3. *The Positive Core of a TU Cooperative Games*

As usual, a transferable utility (TU) cooperative game is represented by the pair (N, a), where a: $2^N \to R$ is the characteristic function. The properties required for a depend on their interpretation. Mostly, one considers $a(S)$ as the maximum total revenue of the players in S, but alternatively we can interpret $a(S)$ as being the minimum total cost (penalty) supported by S.

For this latter case, it is natural and convenient to reverse traditional inequalities in cooperative game theory. Therefore, the core $C(N, a)$ of the game (N,a) will be defined as:

(3) $$C(N,a)=\left\{y\in R^n \mid \sum_{i\in S} y_i \le a(S), \forall S\subset N, \sum y_i = a(N)\right\}$$

By the Bondareva-Shapley Theorem, (see Bondareva, 1963; Shapley, 1967) the core is nonempty if and only if the game is balanced. According to (3), we will say that the game (N,a) is balanced iff, for every balanced family **S**, and for every associated system of weights $(w_S)_{S\in S}$,

$$a(N)\le \sum\nolimits_{S\in S} w_S a(S)$$

Note the following easy property of the core. If the characteristic function a is monotonic, i.e.

(4) $$S\subset T \Rightarrow a(S)\le a(T)$$

then $C(N,a)\subseteq R^n_+$.

In fact, if we consider an $i\in N$ and if $y\in C(N,a)$, then, by the definition of the core, it results:

$$y_i = a(N) - \sum_{j\in N\setminus\{i\}} y_j \ge a(N) - a(N\setminus\{i\}) \ge 0$$

Hence, $y\in R^n_+$.

The above property does not claim the nonemptiness of the core. Condition (4) may come into contradiction with the balancedness of the game. A necessary condition for the nonemptiness of the core of a monotonic game is the positiveness of the characteristic function. Indeed, for any $\varnothing \ne S\subset N$, the balancedness and the monotonicity imply that

$$a(N)\le a(S)+a(\overline{S})\le 2a(N)$$

Hence, $a(N)\ge 0$. Now, if $a(S)<0$, for some S, then from the first inequality it results $a(N)<a(\overline{S})$, contradicting (4).

4. *Existence Result for the Pseudo-core*

The following assumptions will be made for proving the main existence result:

(A1) $\forall\ i \in N$, the demand function d_i is continuous, decreasing and strictly positive with elasticity strictly less than 1, i.e. $|\in| < 1$.

(A2) The total revenue, represented by the scalar product $\langle p, d(p) \rangle$ is strictly increasing on R_+^n.

This assumption is motivated by the elasticity of the demand. If the prices rise then the demand decreases, but the diminution of the demand is slower than the increase of prices, so that total revenue increases.

(A3) The value function v is continuous, non-negative and increasing on R_+^n.

(A4) For every $x \in R_+^n$, every balanced family $S \subseteq 2^N$, and every associated system of weights $(w_s)_{s \in S}$, $v(x) \leq \sum_{S \in S} w_s v(\hat{x}_s)$. The latter inequality shows how agents are motivated to come to a cooperative agreement.

(A5) There exists a positive costant M, such that $\forall p \in R_+^n$, $\frac{v[\hat{d}_i(p)]}{d_i(p)} \leq M$, $\forall i \in N$. In other words the social value of the demand concerning a certain agent is bounded by a linear function of its demand.

Example 2. Consider a *Cournot-Bertrand duopoly* as in Example 1, where the demands are Cobb-Douglas functions:

$$d_1(p_1, p_2) = a \cdot p_1^{-\alpha} \cdot p_2^{-\frac{\alpha}{k}}$$

$$d_2(p_1, p_2) = a \cdot p_1^{-\frac{\alpha}{k}} \cdot p_2^{-\alpha}$$

where $a > 0$, $\alpha \in (0, 1)$ and $k \geq 2$ is an integer.

As reasonable assumption in a cooperative approach, we can consider that the relative prices have a bounded variation. Thus, suppose that $k^{-\frac{2k}{k+1}} < \frac{p_1}{p_2} < k^{\frac{2k}{k+1}}$.

A simple calculation shows that all conditions A1-A5 are verified in this case (note that (A4) is verified with equality).

Theorem 1. *Let G = (N, d, v) satisfy (A1), (A2), (A3), (A4) and (A5), then PC(G) ≠∅.*

The proof of the theorem needs the following discussion. Let be $P_M = \{p \in R_+^n \mid p_i \le M, \forall i \in N\}$ where M is as in (A5). For $p \in P_M$ consider the TU game with the characteristic function a_p defined by $a_p(S) = v[\hat{d}_S(p)], S \subseteq N$. If the demand functions are positive and the value function is increasing, then it follows from (4) that $C(N, a_p) \subseteq R_+^n$.

Now, define the correspondence (set-valued function) φ from P_M to R_+^n by:

$$\varphi(p) = \left\{q \in R_+^n \mid (q_i d_i(p))_{i \in N} \in C(N, a_p)\right\}$$

Lemma 2. *Let G = (N, d, v) satisfy (A1) and (A3)-(A5). Then φ is a closed correspondence with nonempty convex values from P_M to itself. Therefore, it admits a fixed point.*

Proof. By (A4), it follows that the game (N, a_p) is balanced for every $p \in P_M$, so that $C(N, a_p) \ne \varnothing$. As mentioned above, from (A3) it results $C(N, a_p) \subseteq R_+^n$. Hence $\varphi(p) \ne \varnothing$. If $q \in \varphi(p)$, then $\left(q_i d_i(p)\right)_{i \in N} \in C(N, a_p)$, so that $q \ge 0$, and $q_i d_i(p) \le v\left(\hat{d}_i(p)\right), \forall i \in N$ (from the definition of the core). Hence, by (A5) it follows that $q \in P_M$,.

The convexity of $\varphi(p)$ follows from the convexity of the core. Let us show that the correspondence φ is closed.

Consider the convergent sequences $p^k \to p^0, (p^k) \subset P_M$, $q^k \to q^0, q^k \in \varphi(p^k), \forall k = 1, 2, \ldots$ Then, for every k we have:

$$\sum_{i \in S} q_i^k d_i(p^k) \le v[\hat{d}_S(p^k)], \forall S \subset N, \sum_{i \in N} q_i^k d_i(p^k) \le v[\hat{d}(p^k)]$$

From the continuity of d_i and v, it follows that:

$$\sum_{i\in S} q_i^0 d_i(p^0) \leq v[\hat{d}_S(p^0)], \forall S \subset N, \sum_{i\in N} q_i^0 d_i(p^0) \leq v[\hat{d}(p^0)]$$

Since $q^0 \geq 0$, we obtain that $q^0 \in \varphi(p^0)$, which proves the closedness of φ.

Finally, observe that P_M is a compact convex, thus all assumptions of the Kakutani's fixed point theorem are satisfied.

Now we are ready to prove the theorem.

Proof of the Theorem 1. Let p^* be any fixed point of φ. We will show that $p^* \in PC(G)$.

From $p^* \in \varphi(p^*)$ it follows that $p^* \geq 0$ and

$$\left\langle p^*, d(p^*)\right\rangle = \sum_{i\in N} p_i^* d_i(p^*) = v\big(d(p^*)\big)$$

so that (1) is satisfied.

Now, let us assume that there exists some $p \in R_+^n$ verifying (2a) - (2b), for some *S*. Then, by (A_1) and (A_3) we get

$$\left\langle p_S, d_S(p)\right\rangle \geq v\left(\hat{d}_S(p)\right) \geq v\left(\hat{d}_S(p^*)\right)$$

Since $\left(p_i^* d_i(p^*)\right)_{i\in N} \in C(N, a_{p^*})$ it follows that

$$\left\langle p_S^*, d_S(p^*)\right\rangle = \sum_{i\in S} p_i^* d_i p^* \leq v\left(\hat{d}_S\ p^*\right)$$

Hence,

(5) $$\left\langle p_S, d_S(p)\right\rangle \geq \left\langle p_S^*, d_S(p^*)\right\rangle$$

By (A_2) it follows that

$$\langle p, d(p)\rangle < \langle p^*, d(p^*)\rangle$$

By (A_1) it follows that

$$\langle p, d(p)\rangle = \langle p_{\bar{S}}, d_{\bar{S}}(p)\rangle + \langle p_S, d_S(p)\rangle = \langle p^*_{\bar{S}}, d_{\bar{S}}(p)\rangle + \langle p_S, d_S(p)\rangle \geq$$

$$\geq \langle p^*_{\bar{S}}, d_{\bar{S}}(p^*)\rangle + \langle p_S, d_S(p)\rangle$$

Since, obviously,

$$\langle p^*, d(p^*)\rangle = \langle p^*_{\bar{S}}, d_{\bar{S}}(p^*)\rangle + \langle p^*_S, d_S(p^*)\rangle$$

then, the last three relations imply

$$\langle p_S, d_S(p)\rangle < \langle p^*_S, d_S(p^*)\rangle$$

contradicting (5).

5. *Final Comments*

A deeper understanding of the concept of *pseudo-core* could result from the following comments. Let us associate to the *one-commodity market game* $G = (N, d, v)$ the non-transferable utility (NTU) game represented by (N, V), where the coalitional function V is defined by

$$V(S) = \left\{p \in R^n_+ \mid \langle p_S, d_S(p)\rangle \geq v\left(\hat{d}_S(p)\right)\right\}$$

if $\emptyset \neq S \subset N$, and

$$V(N) = \left\{p \in R^n_+ \mid \langle p, d(p)\rangle = v\left(d(p)\right)\right\}$$

Then, we can observe that the *pseudo-core* of G contains the core of (N, V). Thus, in particular, every system of prices belonging to the core of the game (N, V) is stable.

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